Cayley cubic and thin lens formula

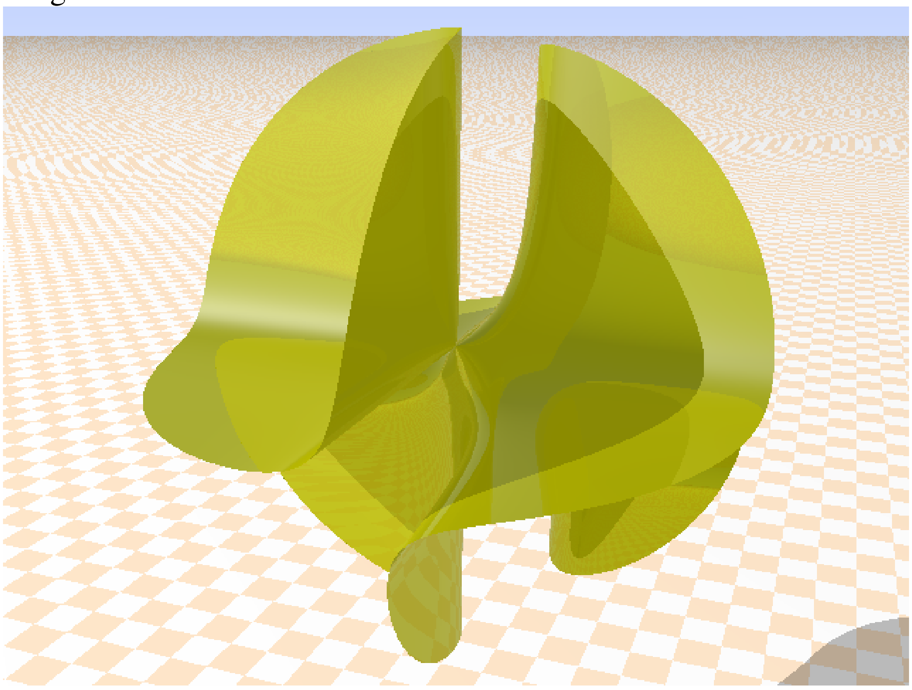
The Cayley cubic can be applied to understand the different possibilities of imaging by thin lenses. In this case of thin lenses the thin lens formula is

$$\frac{1}{v}+\frac{1}{b}=\frac{1}{f}$$

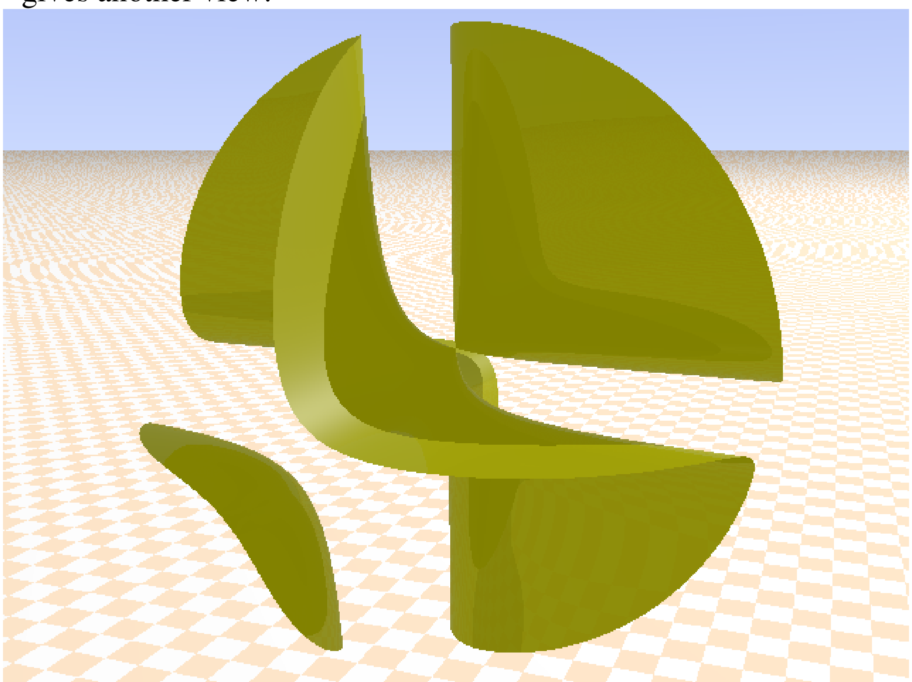
The Cayley cubic can be represented by

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1=0$$

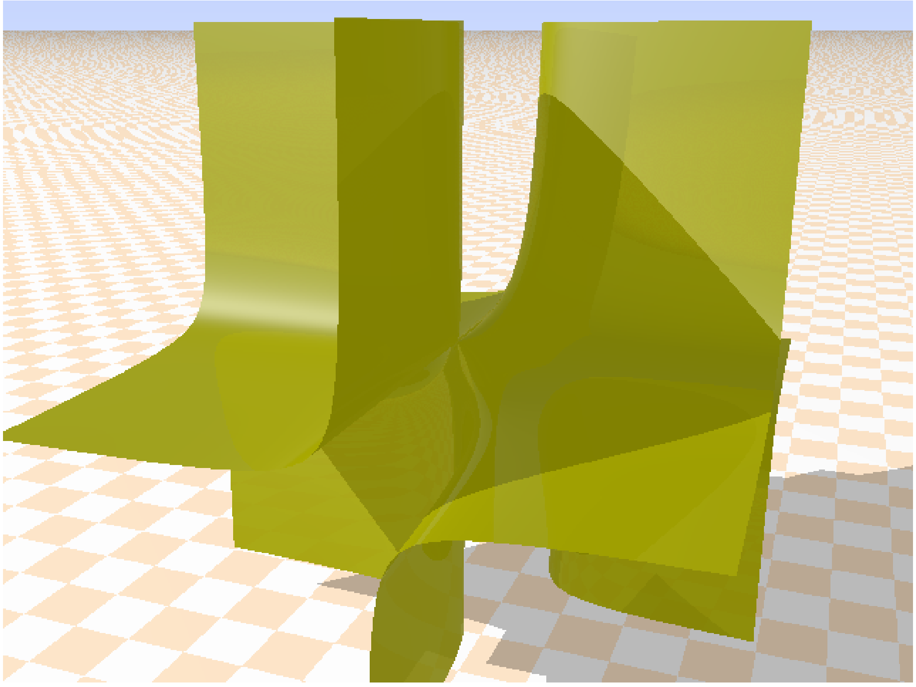
It gets the following form in this affine case:



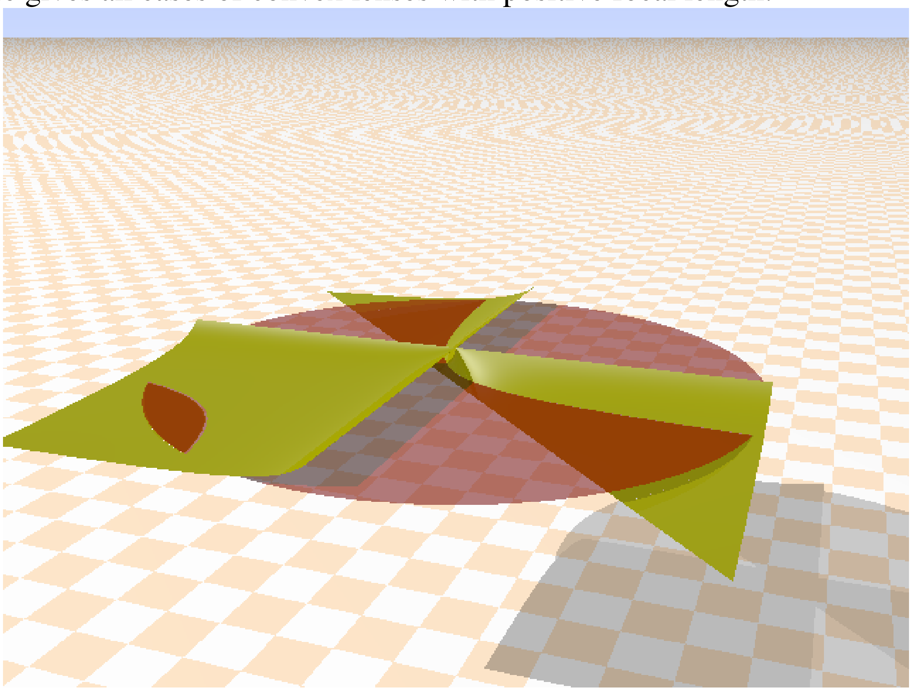
Rotated by 180° gives another view:



The same surface can be limited to a box instead of a sphere:



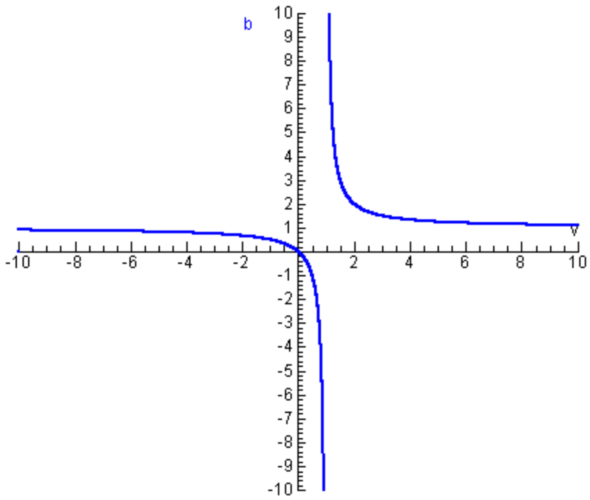
The next picture gives all cases of convex lenses with positive focal length:



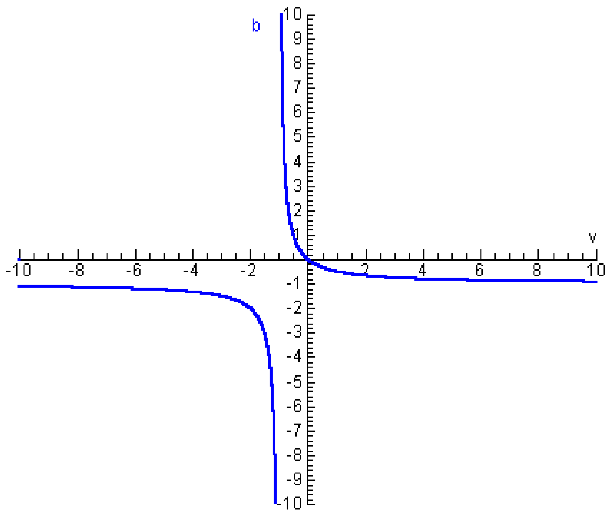
The red plane has equation z=constant where z is the vertical coordinate. Then x and y are the horizontal ones. The relation between z and focal length f is

$$z = \frac{-f}{1+f} \quad \text{or} \quad f = \frac{-z}{1+z}$$

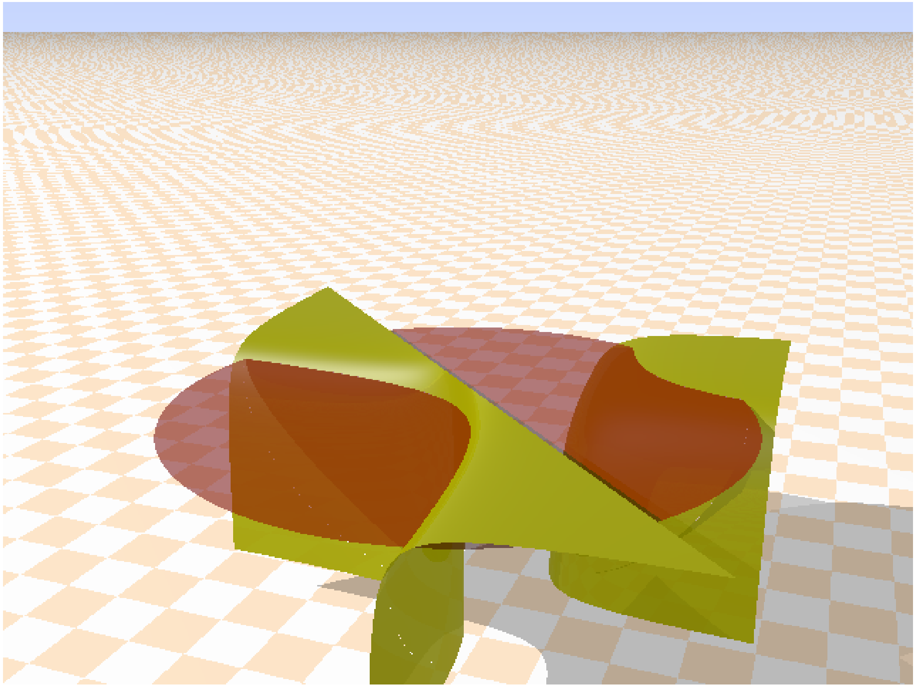
In the picture above the z-coordinate varies between z=0 and z=-1. This covers all positive values of f between 0 and ∞.
In the diagram below the relation between v and b is shown for $f = 1$.
The value of f determines the the asymptotes of the hyperbola. If v → ± ∞ then b → 1. If b → ± ∞ then v → 1.

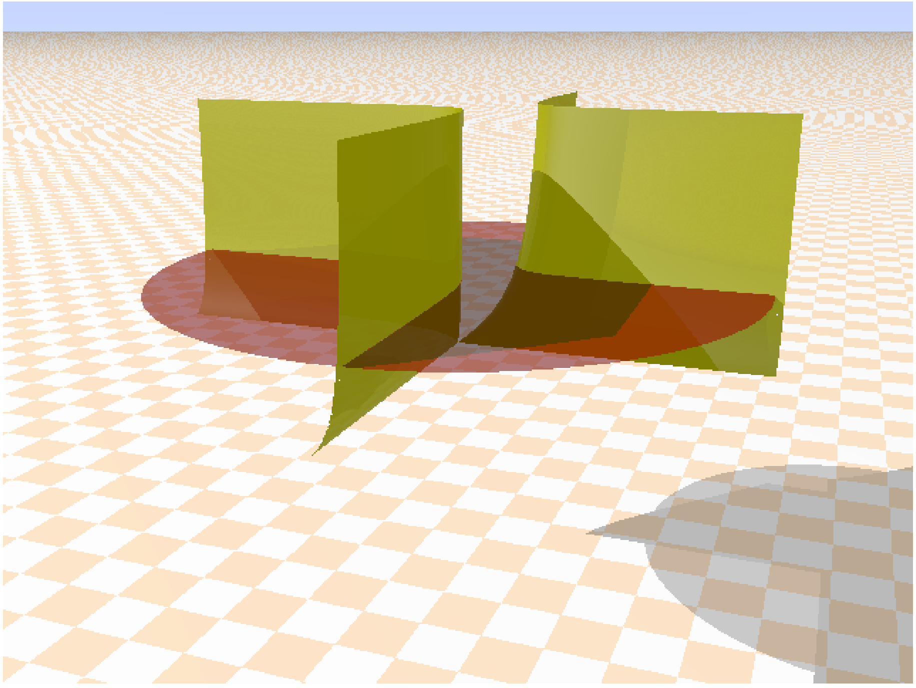


In the diagram below the relation between v and b is shown for $f = -1$.



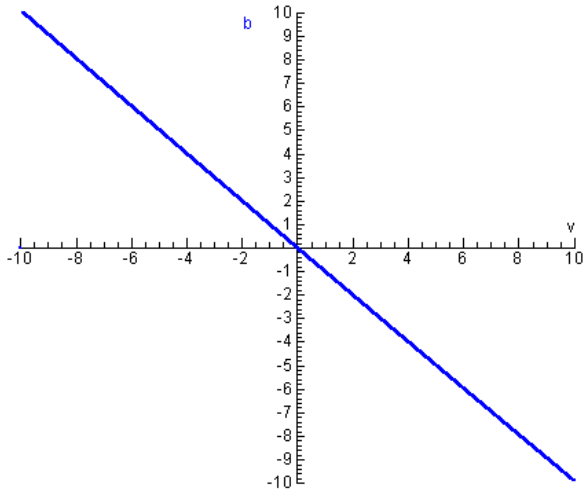
The next 2 pictures gives all cases of concave lenses with negative focal length:





The first picture has all values of z less then –1, where the surface should be extended to z = –∞. This covers all negative values of f between –∞ and –1.

The second picture has all values of z larger then 0, where the surface should be extended to z = ∞. These cover all negative values of f between 0 and –1.
If f = 0 the lens formula tells us that b or v should be zero. So the 2 coordinate axes in the (z = 0) – plane, x = 0 and y = 0, represent b = 0 and v = 0.
The straight line at z = –1 ($f = \pm\infty$) is a degenerated branch of a hyperbola, where the other branch is a straight line at infinity that is parallel to it. This the case of a virtual image of a flat window pane coinciding with the object.



In all other cases the horizontal plane cuts out two branches of a hyperbola.