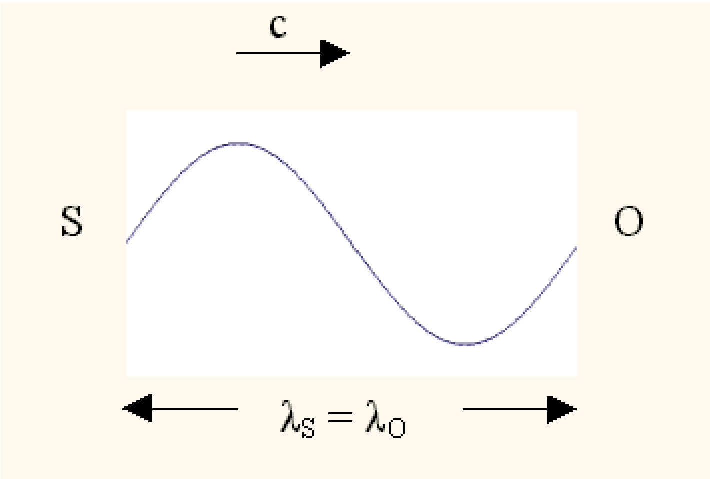
The relativistic Doppler effect

According to the special relativity theory two principles must be satisfied:

1) There is no preferred position in space that can be called at rest. Only relative speed has physical meaning.

2) The velocity of light c is always the same to each observer.

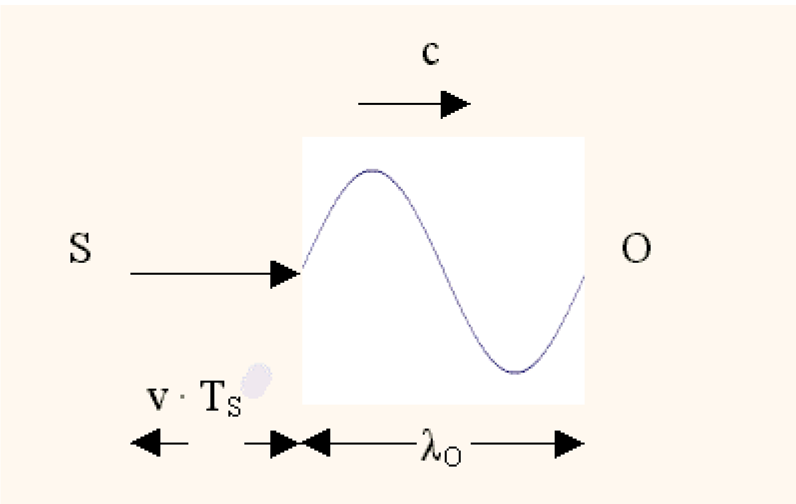
Let's look at a light source S that has no speed relative to an observer O.



The observer O receives a light wave from the source S. The wave length of the emitted wave is λ_S. This wavelength is equal to the observed wave length λ_O. With the velocity c and period T we can write $\lambda = c \cdot T$ in general and in the present case:

$\lambda_S = c \cdot T_S$ and $\lambda_O = c \cdot T_O$

Now suppose the source is moving with velocity v in the direction of the observer. Let T_S be the time in which one wavelength is emitted as measured by a clock that is moving along with S, viewed in a coordinate frame where O is at rest .



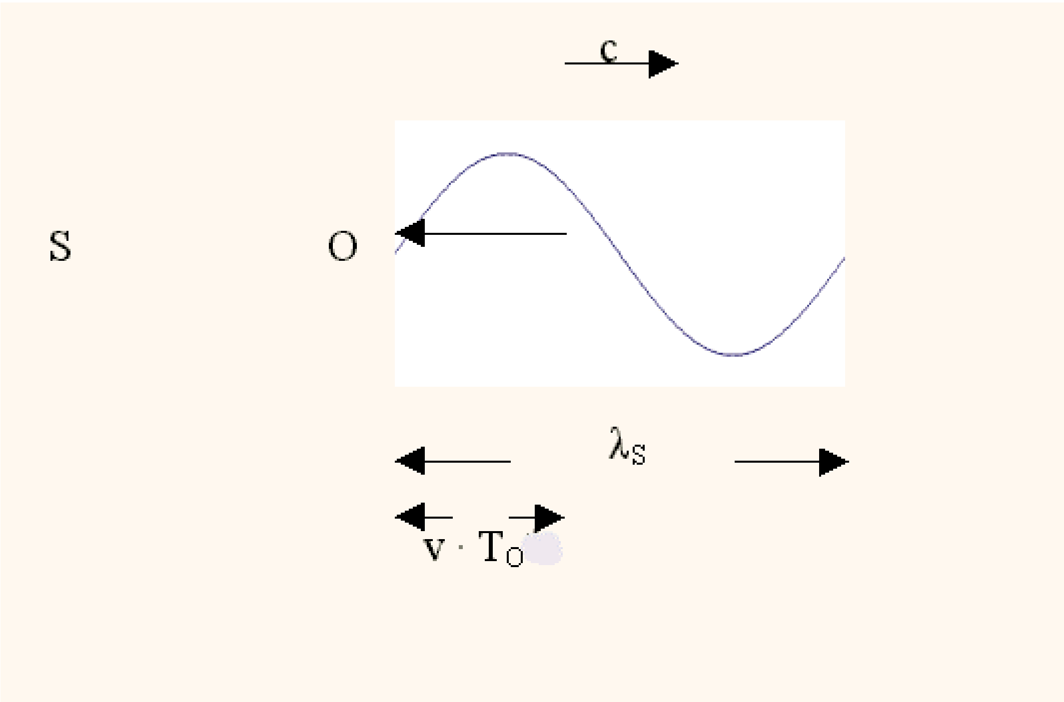
We see that the observed wavelength is shorter $\lambda_O = \lambda_S - v \cdot T_S$.

From this we can get $c \cdot T_O = c \cdot T_S - v \cdot T_S$.

Thus the observed period in case of a moving source is

$$T_O = T_S \cdot \frac{c - v}{c} \qquad (1)$$

Now let's suppose that the source is at rest and the observer is moving with velocity v in the direction of the source. Let T_O be the time in which the observer passes one wavelength, as measured by a clock that is moving along with the observer.



In the time T_O the observer travels a distance $v \cdot T_O$ to the left and the light wave travels a distance $\lambda_S - v \cdot T_O$ to the right. The light's distance is also equal to $c \cdot T_O$.

So $\lambda_S - v \cdot T_O = c \cdot T_O$.

Or $c \cdot T_S = c \cdot T_O + v \cdot T_O$.

The observed period in case of a moving observer is

$$T_O = T_S \cdot \frac{c}{c + v} \qquad (2)$$

Let's call the period when measured in the rest frame of the observer T_O^*. Then

$$T_O^* \cdot \left(\frac{T_O}{T_O^*} \right) = T_S \cdot \frac{c}{c+v}$$

or

$$\frac{T_O^*}{T_S} = \frac{T_O^*}{T_O} \cdot \frac{c}{c+v}$$

This result is to be compared with the case of a moving source (1) : $T_O = T_S \cdot \frac{c-v}{c}$

Let's call the period when measured in the rest frame of the source T_S*. Then

$$T_O = \left(\frac{T_S}{T_S^*} \right) \cdot T_S^* \cdot \frac{c-v}{c}$$

$$\frac{T_O}{T_S^*} = \left(\frac{T_S}{T_S^*} \right) \cdot \frac{c-v}{c}$$

According to principle of relativity of motion the effect on the period should only depend on the relative velocity v and should be independent of having the first case of a moving source or the second case of a moving observer. So we should have only one value for T_O*/ T_S = T_O / T_S*.

$$\left(\frac{T_O^*}{T_O} \right) \cdot \frac{c}{c+v} = \left(\frac{T_S}{T_S^*} \right) \cdot \frac{c-v}{c}$$

Or

$$\left(\frac{T_O^*}{T_O} \right) \cdot \left(\frac{T_S^*}{T_S} \right) = \frac{c^2 - v^2}{c^2}$$

Then let's suppose that the effect of motion is the same for all clocks:

$$\left(\frac{T^*}{T} \right)^2 = \frac{c^2 - v^2}{c^2}$$

The ratio of times measured by clocks at rest and times measured by moving clocks is then given by

$$\frac{T^*}{T} = \sqrt{\frac{c^2 - v^2}{c^2}} = \sqrt{1 - v^2 / c^2} \qquad (3)$$

Moving clocks appear to run to slower for an observer at rest. For moving observers clocks at rest appear to run slower. The period T* in the rest frame of the clock is shortest.

For the longitudinal (i.e. directions of v and c on the same line) relativistic Doppler effect we get

$$T_O = \frac{T_S^*}{\sqrt{1-v^2/c^2}} \cdot \frac{c-v}{c} = T_S^* \cdot \sqrt{\frac{c-v}{c+v}} \tag{4}$$

in case of the moving source.

And the same result in case of the moving observer

$$T_O^* = \left(T_S \cdot \frac{c}{c+v}\right) \cdot \sqrt{1-v^2/c^2} = T_S \cdot \sqrt{\frac{c-v}{c+v}} \tag{5}$$

If the observer and the source approach each other by a relative velocity v, the observed period T_O becomes smaller. This results in higher observed frequency f_O .

We can write (5) as

$$T_O^* = T_S \cdot \frac{1-v/c}{\sqrt{1-v^2/c^2}}$$

$$T_O^* = \frac{T_S - \lambda_S \cdot v/c^2}{\sqrt{1-v^2/c^2}} \tag{6}$$

The time T_S is corrected by the time that the light needs to travel the distance $v \cdot T_S$. The time T_S is measured by two different clocks which are separated by the distance .Expression (6) is the Lorentz transformation for the time coordinate from one coordinate frame (attached to the source) to another (attached to the observer) that moves with relative speed v. The general form is:

$$t^* = \frac{t - x \cdot v/c^2}{\sqrt{1-v^2/c^2}} \tag{7}$$

Multiplying (6) by c we get

$$\lambda_O^* = \frac{\lambda_S - v \cdot T_S}{\sqrt{1-v^2/c^2}} \tag{8}$$

The distance is corrected by the distance $v \cdot T_S$. Expression (8) is the Lorentz transformation for the space coordinate. The general form is:

$$x^* = \frac{x - v \cdot t}{\sqrt{1-v^2/c^2}} \tag{9}$$