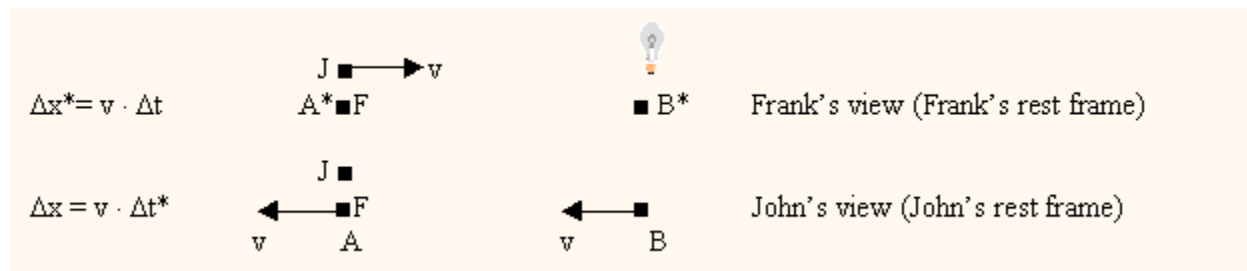


# Special relativity theory

In relativistic kinematics there are two basic assumptions:

1. The speed of light (in empty space) is always the same,  $c$  say.
2. Motion is relative, i.e. it makes no difference if (the frame of) an observer moves with respect to the (frame of the) object or the (frame of) the object moves with respect to the (frame of the) observer .



In Frank's view John is moving from  $A^*$  to  $B^*$  with speed  $v$ . Frank calls the distance  $A^*B^*$  *proper length*  $\Delta x^*$  and measures the time of travel on the moving clock (John's watch) as  $\Delta t$ .

John is at rest in his view and thinks that  $AB$  (with Frank) is moving with speed  $-v$ . The time for  $AB$  to pass by (on his watch) he calls *proper time*  $\Delta t^*$ . He measures the distance of the moving distance  $AB$  as  $\Delta x$ .

When John arrives at  $B$ , he looks at  $A$  by means of the light of a bulb at  $B$ . The light that falls in his eye has, in Frank's view, travelled the distance  $2 \cdot \Delta x^*$  in a time, measured by a clock resting at  $B$ ,

$$\Delta t_{light}^* = \Delta t_{light,1}^* + \Delta t_{light,2}^* = \frac{\Delta x^*}{c} + \frac{\Delta x^*}{c} = 2 \frac{\Delta x^*}{c} = 2v \frac{\Delta t}{c} \quad (1)$$

where  $c$  is the speed of light.

John in his view stays where he is and waits until  $B$  passes by. Then he looks at  $A$ . The light that falls in his eye has travelled in a time  $\Delta t_{light,2}$ , measured by a moving clock at  $B$ , from the position of point  $A$  at the time  $\Delta t^* - \Delta t_{light,2}$  to the eye of  $F$ . This distance is  $\Delta x - v \cdot \Delta t_{light,2} = v \cdot (\Delta t^* - \Delta t_{light,2})$ . This distance can also be computed as  $c \cdot \Delta t_{light,2}$ .

This results, on equalizing these two expressions for the travelling distance of the light pulse, in:

$$\Delta t_{light,2} = \Delta t^* \frac{v}{c + v} \quad (2)$$

Before arriving at  $A$  the light pulse had to travel from bulb  $B$  to  $A$  in a time  $\Delta t_{light,1}$ . The distance to travel for the light pulse is  $\Delta x + v \cdot \Delta t_{light,1} = v \cdot (\Delta t^* + \Delta t_{light,1})$ . This distance is also equal to  $c \cdot \Delta t_{light,1}$ .

This results in an expression for  $\Delta t_{light,1}$  :

$$\Delta t_{light,1} = \Delta t^* \frac{v}{c - v} \quad (3)$$

The time for the light pulse to return at B is

$$\Delta t_{light} = \Delta t_{light,1} + \Delta t_{light,2} = \Delta t^* \frac{v \cdot 2c}{c^2 - v^2} \quad (4)$$

Now

$$\frac{\Delta t_{light}^*}{\Delta t_{light}} = \frac{\Delta t^*}{\Delta t} \left( \frac{1}{1 - v^2/c^2} \right) \quad \text{or} \quad \frac{\Delta t_{light}^*}{\Delta t_{light}} \cdot \frac{\Delta t^*}{\Delta t} = \frac{1}{1 - v^2/c^2}$$

and if we assume the effect on all clocks to be the same, this suggests for times  $\Delta t$  of moving clocks and times  $\Delta t^*$  of clocks at rest the general relation

$$\frac{\Delta t^*}{\Delta t} = \sqrt{1 - v^2/c^2} \quad (5)$$

Moving clocks run slower. This is called *time dilation*.

As  $v = \Delta x / \Delta t^* = \Delta x^* / \Delta t$  we get a relation between distances

$$\frac{\Delta x}{\Delta x^*} = \frac{\Delta t^*}{\Delta t} \quad \text{or} \quad \frac{\Delta x}{\Delta x^*} = \sqrt{1 - v^2/c^2} \quad (6)$$

This is called the *Lorentz contraction*. Moving objects become shorter in the direction of motion.

Recall that  $\Delta x^*$  is measured by synchronized clocks ("observed simultaneously") and the light pulse that travels from B\* to A\* and back meets the eye when the observer is at B\* (A\* and B\* are observed simultaneously).

A further result can be achieved:

$$\frac{\Delta t_{light,1}^*}{\Delta t_{light,1}} = \frac{v \Delta t / c}{\Delta t^* \frac{v}{c - v}} = (1 - v/c) \frac{\Delta t}{\Delta t^*} \quad \text{or} \quad \frac{\Delta t_{light,1}^*}{\Delta t_{light,1}} = \sqrt{\frac{c - v}{c + v}} \quad (7)$$

and

$$\frac{\Delta t_{light,2}^*}{\Delta t_{light,2}} = \frac{v \Delta t / c}{\Delta t^* \frac{v}{c + v}} = (1 + v/c) \frac{\Delta t}{\Delta t^*} \quad \text{or} \quad \frac{\Delta t_{light,2}^*}{\Delta t_{light,2}} = \sqrt{\frac{c + v}{c - v}} \quad (8)$$

This is the *longitudinal Doppler effect for light*. To see this suppose that in  $\Delta t_{\text{light},1}$  the source makes  $n$  oscillations of period  $T_s$ :

$$\Delta t_{\text{light},1} = n \cdot T_s.$$

Let  $\Delta t_{\text{light},1}^*$  be the time in which the observer measures  $n$  oscillation periods  $T_0^*$ :

$$\Delta t_{\text{light},1}^* = n \cdot T_0^*.$$

Then (7) becomes

$$\frac{T_0^*}{T_s} = \sqrt{\frac{c - v}{c + v}}$$

in case the source and the observer approach each other.

In the same way (8) becomes

$$\frac{T_0^*}{T_s} = \sqrt{\frac{c + v}{c - v}}$$

in case the source and the observer are removing of each other.

The first equation gives the effect of a light source (the bulb) that moves toward the observer. The second gives the effect of a light source (the reflecting point A) that moves away from the observer. The effect is the same for a moving observer and light source at rest, because you must change stars and plus and minus signs simultaneously. In John's view the bulb moves with speed  $-v$  and in Frank's view the speed of John is  $v$ .

To understand why the ratio  $\Delta t_{\text{light},1}^* / \Delta t_{\text{light},1}$  differs from  $\Delta t_{\text{light}}^* / \Delta t_{\text{light}}$  we must realize that the latter times are measured by one clock and the former by two clocks at different positions. We can write:

$$\frac{\Delta t_{\text{light},1}^*}{\Delta t_{\text{light},1}} = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}$$

This can be rewritten (using  $c \cdot \Delta t_{\text{light},1} = \Delta x_{\text{light},1}$  for the distance between the clocks) as

$$\Delta t_{\text{light},1}^* = \frac{\Delta t_{\text{light},1} - v \cdot \Delta x_{\text{light},1} / c^2}{\sqrt{1 - v^2/c^2}}$$

This is the *Lorentz transformation for the time coordinate*. In general:

$$\Delta t^* = \frac{(\Delta t - v \cdot \Delta x / c^2)}{\sqrt{1 - v^2/c^2}} \quad (9)$$

It shows that two clocks that are separated by a distance  $\Delta x$  in one frame measure an extra difference

$$\Delta t^* - \frac{\Delta t}{\sqrt{1 - v^2/c^2}} = - \frac{v \cdot \Delta x / c^2}{\sqrt{1 - v^2/c^2}}$$

in the other frame moving with relative speed  $v$ . This is called the *phase difference*.

It means that even if in one frame there is no difference in time ( $\Delta t = 0$ ), in the other one there is a difference

$$\Delta t^* = - \frac{v \cdot \Delta x / c^2}{\sqrt{1 - v^2/c^2}}$$

Simultaneity depends on the frame you're in.

Less strange is the result for the space coordinate. From

$$\frac{\Delta t_{light,1}^*}{\Delta t_{light,1}} = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad \Delta t_{light,1} = \frac{\Delta x_{light,1}}{c} \quad \text{we get}$$

$$c \cdot \Delta t_{light,1}^* = \frac{c \cdot \Delta t_{light,1} - v \cdot \Delta x_{light,1}}{\sqrt{1 - v^2/c^2}}$$

or

$$\Delta x_{light,1}^* = \frac{\Delta x_{light,1} - v \cdot \Delta t_{light,1}}{\sqrt{1 - v^2/c^2}}$$

This is the *Lorentz transformation for the space coordinate*. In general:

$$\Delta x^* = \frac{\Delta x - v \Delta t}{\sqrt{1 - v^2/c^2}} \quad (10)$$

Adding of velocities is easily computed with the aid of the Lorentz transformations of space and time coordinate.

$$\text{With (10)} \quad \Delta x^* = \frac{\Delta x - v \Delta t}{\sqrt{1 - v^2/c^2}}$$

$$\text{And (9)} \quad \Delta t^* = \frac{(\Delta t - v \cdot \Delta x / c^2)}{\sqrt{1 - v^2/c^2}}$$

we get

$$\frac{\Delta x^*}{\Delta t^*} = \frac{\Delta x - v \cdot \Delta t}{\Delta t - v \cdot \Delta x / c^2} = \frac{\Delta x / \Delta t - v}{1 - v \cdot (\Delta x / \Delta t) / c^2}$$

Call  $\Delta x^* / \Delta t^* = u^*$  and  $\Delta x / \Delta t = u$  then

$$u^* = \frac{u - v}{1 - uv/c^2} \quad (11)$$

and , solving u ,

$$u = \frac{u^* + v}{1 + u^* v / c^2} \quad (12)$$

To compute the acceleration  $a^* = du^*/dt^*$  if  $a = du/dt$  is known, we start with

$$du^* = \frac{du \cdot (1 - u \cdot v / c^2) + (u - v) \cdot du \cdot v / c^2}{(1 - u \cdot v / c^2)^2} = \frac{du \cdot (1 - v^2 / c^2)}{(1 - u \cdot v / c^2)^2}$$

and

$$dt^* = \frac{dt - v \cdot dx / c^2}{\sqrt{1 - v^2 / c^2}} \quad \text{and} \quad \frac{dt^*}{dt} = \frac{1 - u \cdot v / c^2}{\sqrt{1 - v^2 / c^2}}$$

Dividing, we get

$$a^* = a \cdot \left( \frac{(1 - v^2 / c^2)^{3/2}}{(1 - uv / c^2)^3} \right) \quad (13)$$

In a frame where the object has a velocity  $u^* = 0$  (its rest frame) (then  $u = v$ ) we get

$$a_0^* = \frac{a_0}{(1 - v^2 / c^2)^{3/2}} \quad (14)$$